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INTERLAMINAR STRESS ANALYSIS OF COMPOSITE LAMINATES USING A SUBLAMINATE/LAYER MODEL†

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Abstract—A stress-function-based variational method developed in a previous work is extended and modified into a sublaminate/layer approach applicable to a laminated strip composed of a large number of differently orientated, anisotropic elastic plies. Lekhnitskii's stress functions are used for two interior layers adjacent to a particular interface. The remaining layers are grouped into an upper sublaminate and a lower sublaminate. The stress functions are expanded in truncated power series of the thickness coordinate and the differential equations governing the coefficient functions are derived by using the complementary virtual work principle. The new approach limits the dimension of the eigenvalue problem to a fixed number irrespective of the number of layers in the sublaminates, so that reasonably accurate solutions of the interlaminar stresses can be computed with extreme ease. For symmetric, four-layer, angle-ply and cross-ply laminates, a comparison of the previous analysis results based on the pure layer model and new results based on two different sublaminate/layer models indicates reasonable overall agreement in the interlaminar stresses and superior agreement in the resultant peeling and shearing forces across end segments of the interface. Additional results are obtained for eight-layer, quasi-isotropic laminates under the strain loads of axial extension, bending and twisting.

1. INTRODUCTION

In a previous work (Yin, 1991), a stress-based variational method was developed for determining the interlaminar stresses in a multi-layered strip with two parallel free edges, $x = \pm a$, when the strip is subjected to an extensional strain ε_z parallel to the free edges, a bending curvature κ_z, or a twisting deformation κ_{xz} [see Pagano and Soni (1989) for selected references to the large body of previous analytical and computational works on the problem, mostly for the case of an axial strain load]. In an interior segment of the strip away from the two end regions, the stresses in each anisotropic layer are independent of the axial coordinate z. Consequently, the displacement functions in each layer correspond to a generalized plane deformation (Lekhnitskii, 1963), and the stresses in the layer may be expressed as the derivatives of a pair of Lekhnitskii's stress functions $F(x, y)$ and $\Psi(x, y)$. Using polynomial expansions of the stress functions with respect to the thickness coordinate y, and the continuity conditions of the interlaminar stresses across the layer interfaces, variational equations associated with the principle of stationary complementary energy may be derived for a set of functions $\{X_i(x)\}$, which are the values of the stress functions and their y-derivatives on the interfaces. These variational (Euler–Lagrange) equations are ordinary differential equations with constant coefficients. Together with the homogeneous boundary conditions for X_i at the free edges $x = \pm a$, the equations define an eigenvalue problem whose solutions determine the stress functions (and hence also the stresses) in all layers of the laminated strip.

An important feature of the stress-function-based variational method is that the admissible stress fields satisfy *exactly* the equilibrium equations in each layer, including the corner regions surrounding the intersection of a free edge with an interface, where the stresses and the stress gradients may be exceedingly large. Furthermore, boundary and interface continuity conditions are imposed upon the stress functions in such a way that

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the admissible stress fields also satisfy *exactly* the traction boundary conditions and the continuity of interlaminar stresses along entire interfaces, including the end segments with severe stresses. These properties of the admissible stress fields contribute to the overall superior accuracy of the resulting stress solution (selected from the class of admissible stress fields by the stationary complementary work principle) compared to the stress field calculated from the displacement solution of a displacement-based variational analysis. The complementary virtual work principle, which yields the system of differential equations for the functions X_i and certain natural conditions along the interfaces, ensures the compatibility of strain and the interfacial continuity of tangential displacements in an averaged sense.

In principle, the stress-function-based variational method is applicable to a laminated strip composed of unidirectional homogeneous layers with arbitrary anisotropic elastic properties and orientation angles. Furthermore, the accuracy of the resulting solution for the interlaminar stresses may be improved by using higher-degree polynomials to approximate the stress functions in each layer (Yin, 1992b). This increases the number of undetermined functions in the set $\{X_i\}$ and, consequently, raises the dimension of the eigenvalue problem. The latter also increases with the total number of layers in the laminate.

Laminates used in aerospace applications are often composed of a relatively large number of thin layers. If the layers are thin, a cubic polynomial expansion of the stress function F and a quadratic expansion of Ψ (as used in Yin, 1991), may yield sufficiently accurate overall patterns of the interlaminar stresses. However, with a large number of layers, the size of the eigenvalue problem may become too great so that accurate computation of *all* eigenvalues and eigenfunctions is unduly laborious. The principal advantage of the present method, its ability to yield accurate interlaminar stresses through efficient numerical computation, may then be lost.

In the present paper, a modified analysis method based on a sublaminate/layer model is developed. The model is composed of two interior layers adjacent to a particular interface on which the interlaminar stresses are to be determined, and two sublaminates above and beneath these two layers. The layers are considered as homogeneous anisotropic or orthotropic elastic media, while the sublaminates are considered as anisotropic laminates. It should be mentioned that the existence and nature of the stress singularity at the intersection of an interface with a free edge is determined essentially by the mismatch of the elastic moduli of the two layers adjacent to the interface. Consequently, the anisotropic elastic properties of the two layers are of fundamental importance to the local stress field, and these properties should be properly included in any analytical model intended for an accurate prediction of the interlaminar stresses. The remaining layers in the laminate, which are not adjacent to the interface, produce indirect effects that may be evaluated by an approximate analysis in which the layers are grouped into two sublaminates.

In order to obtain a purely stress formulation for the sublaminate/layer model, one must express all kinematical variables of the sublaminates in terms of the stress functions in the two interior layers. This task is achieved in the next section. First, the stiffness matrices of the sublaminate are used to express the kinematical variables in terms of the stress and moment resultants in the sublaminate. Then the equilibrium equations of the sublaminate are integrated to obtain the stress and moment resultants in terms of the values of the layer stress functions and their normal derivatives on the sublaminate/layer interface. These two sets of relations are subsequently used to eliminate the kinematical variables from the complementary virtual work principle of the sublaminate/layer model, so that the resulting equation depends only on the stress functions in the two interior layers (Section 3). Since the elimination of the sublaminate kinematical variables is achieved by using the equilibrium equations of the sublaminates, no additional requirements need be further imposed on the stress functions to ensure its statical admissibility in the variational problem of the sublaminate/layer model.

In Section 4, polynomial expansions of the stress functions in the two interior layers are introduced and the complementary virtual work princple is used to derive the Euler–Lagrange equations governing the coefficient functions of the expansions. The derivation requires complex algebraic manipulations involving the geometrical, elastic and loading parameters. This task is achieved once and for all by using the symbolic algebraic program

MACSYMA (1988). The results are incorporated in a computer program for formulating and solving the eigenvalue problem and obtaining the interlaminar stresses.

In Section 5, two different sublaminate/layer models are applied to the classical problem of a symmetric, four-layer, cross-ply or $\pm 45°$ angle-ply laminate. The interlaminar stresses on the highest interface associated with each one of the three distinct cases of strain loads (axial extension, bending and twisting) are compared for the two models, and also compared with the previous variational solutions of the four-layer laminate [Yin (1991), where no sublaminate was used]. Reasonable agreement among the results of all three analytical models is found, especially for the dominant component of the interlaminar stress. Finally, we apply the sublaminate/layer model to two types of symmetric, eight-layer, quasi-isotropic laminates, which have been previously studied by Wang and Crossman (1977) using the finite element method for the case of axial strain load only. In the present analysis, interlaminar stresses associated with all three distinct strain loads are determined on the various interfaces.

2. THE SUBLAMINATE/LAYER MODEL

A sublaminate/layer model of a multi-layered laminate is shown in Fig. 1. The four parts of this model (the lower sublaminate, the lower and upper interior layers, and the upper sublaminate) are separated by three consecutive interfaces having the thickness coordinates y_1, y_2 and y_3, where the coordinate plane, $y = 0$, is chosen to be the middle plane between the upper and lower surfaces of the laminate, $y = \pm t/2$.

The elastic properties of the two interior layers immediately below and above the interface $y = y_2$ may be characterized by their respective elastic compliance matrices, i.e. $[\underline{a}_{ij}]$ and $[\bar{a}_{ij}]$. In the lower layer, one has

$$\begin{Bmatrix} \sigma_z \\ \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{bmatrix} 1/\underline{a}_{11} & -\underline{a}_{12}/\underline{a}_{11} & -\underline{a}_{13}/\underline{a}_{11} & 0 & 0 & -\underline{a}_{16}/\underline{a}_{11} \\ \underline{a}_{12}/\underline{a}_{11} & \underline{\beta}_{22} & \underline{\beta}_{23} & 0 & 0 & \underline{\beta}_{26} \\ \underline{a}_{13}/\underline{a}_{11} & \underline{\beta}_{23} & \underline{\beta}_{33} & 0 & 0 & \underline{\beta}_{36} \\ 0 & 0 & 0 & \underline{\beta}_{44} & \underline{\beta}_{45} & 0 \\ 0 & 0 & 0 & \underline{\beta}_{45} & \underline{\beta}_{55} & 0 \\ \underline{a}_{16}/\underline{a}_{11} & \underline{\beta}_{26} & \underline{\beta}_{36} & 0 & 0 & \underline{\beta}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_z \\ \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}, \tag{1}$$

where

$$\underline{\beta}_{ij} = \underline{a}_{ij} - \underline{a}_{1i}\underline{a}_{1j}/\underline{a}_{11}, \quad \text{for } i,j \neq 1. \tag{2}$$

Similar relations involving the compliance coefficients $[\bar{a}_{ij}]$ and $[\bar{\beta}_{ij}]$ are valid in the upper layer, where

$$\bar{\beta}_{ij} \equiv \bar{a}_{ij} - \bar{a}_{1i}\bar{a}_{1j}/\bar{a}_{11}, \quad \text{for } i,j \neq 1. \tag{3}$$

We introduce a pair of stress functions $F(x,y)$ and $\Psi(x,y)$ in the two interior layers such that (where the subscripts following a comma indicate partial differentiation with respect to the coordinate variables)

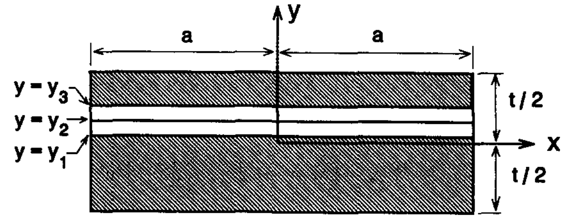

Fig. 1. Sublaminate/layer model for determining the interlaminar stresses on the interface $y = y_2$.

$$\sigma_x = F_{,yy}, \quad \sigma_y = F_{,xx}, \quad \tau_{xy} = -F_{,xy}, \quad \tau_{xz} = \Psi_{,y}, \quad \tau_{yz} = -\Psi_{,x}. \tag{4}$$

The preceding expressions ensure that the stresses in each layer satisfy the equilibrium equations identically. One may choose the stress functions in the two layers in such a way that Ψ, F and $F_{,x}$ vanish on the free edges. One may further require, without loss of generality, that Ψ, F and $F_{,y}$ are continuous across the interface of these layers (Yin, 1991). The values of Ψ, F and $F_{,y}$ on the interfaces $y = y_i$ will be denoted, respectively by $\Psi_i(x)$, $F_i(x)$ and $G_i(x)$ $(i = 1, 2, 3)$.

The displacements of the lower layer are given by (Lekhnitskii, 1963)

$$\begin{aligned}
\underline{u}(x, y, z) &= -\Theta yz + \underline{U}(x, y) + \underline{\omega}_2 z - \underline{\omega}_3 y + \underline{u}_0 \\
\underline{v}(x, y, z) &= \kappa_z z^2/2 + \Theta xz + \underline{V}(x, y) + \underline{\omega}_3 x - \underline{\omega}_1 z + \underline{v}_0 \\
\underline{w}(x, y, z) &= (\varepsilon_0 - \kappa_z y)z + \underline{W}(x, y) + \underline{\omega}_1 y - \underline{\omega}_2 x + \underline{\omega}_0.
\end{aligned} \tag{5}$$

Similar expressions may be written for the displacements of the upper layer. In order to ensure the continuity of the displacements across the interface $y = y_2$, the displacement functions of the two layers must contain the same constants ε_0, κ_z and Θ, which characterize, respectively, the strain loads of extension, bending and twisting.

For the sake of simplicity we shall consider the upper and lower sublaminates as classical laminated plates whose elastic properties are characterized by the extensional, bending and coupling stiffness matrices, i.e. $[\underline{A}_{ij}]$, $[\underline{D}_{ij}]$ and $[\underline{B}_{ij}]$ for the lower sublaminate and $[\bar{A}_{ij}]$, $[\bar{B}_{ij}]$ and $[\bar{D}_{ij}]$ for the upper sublaminate. The use of more refined (higher order) sublaminate models may yield improved analytical results for the interlaminar stresses in comparison with the present solutions. However, since the free edge condition implies that the integral of τ_{xy} across the thickness of each layer vanishes at the free edge, and the integral across the entire laminate thickness vanishes for all x, transverse shear effect is expected to be of minor importance in the present problem.

Let $\underline{H} = t/2 + y_1$ denote the thickness of the lower sublaminate. Then the in-plane strains $\varepsilon_{x(1)}$, $\varepsilon_{z(1)}$ and $\gamma_{xz(1)}$ on the interface $y = y_1$ are related to the same strains on the middle plane of the lower sublaminate, $\underline{\varepsilon}_x^0$, $\underline{\varepsilon}_z^0 = \varepsilon_0 + \kappa_z(t - \underline{H})/2$ and $\underline{\gamma}_{xz}^0$, according to

$$\begin{aligned}
\varepsilon_{x(1)} &= \underline{\varepsilon}_x^0 - \underline{\kappa}_x \underline{H}/2, \qquad \gamma_{xz(1)} = \underline{\gamma}_{xz}^0 - \Theta \underline{H} \\
\varepsilon_{z(1)} &= \underline{\varepsilon}_z^0 - \kappa_z \underline{H}/2 = \varepsilon_0 + (t/2 - \underline{H})\kappa_z,
\end{aligned} \tag{6}$$

where $\underline{\kappa}_x$ is the curvature of the lower sublaminate in the x-direction.

Let $\underline{N}_x$, $\underline{Q}_x$ and $\underline{M}_x$ denote, respectively, the normal force, the shearing force (in the thickness direction) and the bending moment acting on a cross section (x = constant) of the lower sublaminate. Let $\underline{N}_{xz}$ denote the in-plane shearing force in the sublaminate. These stress and moment resultants satisfy the following equilibrium equations

$$\begin{aligned}
\underline{N}_{x,x} + \tau_{xy(1)} &= 0, \qquad \underline{N}_{xz,x} + \tau_{yz(1)} = 0, \\
\underline{Q}_{x,x} + \sigma_{y(1)} &= 0, \quad \underline{M}_{x,x} + \underline{Q}_x - \tau_{xy(1)}\underline{H}/2 = 0,
\end{aligned} \tag{7}$$

where the interlaminar stresses $\sigma_{y(1)}$, $\tau_{xy(1)}$ and $\tau_{yz(1)}$ on the lower interface may be expressed in terms of the derivatives of $F_1(x)$, $G_1(x)$ and $\Psi_1(x)$

$$\sigma_{y(1)} = F_1'', \quad \tau_{xy(1)} = -G_1', \quad \tau_{yz(1)} = -\Psi_1'.$$

Integrating the preceding equations with respect to the coordinate x, and using the free edge condition, one obtains the following results

$$\underline{N}_x = G_1, \quad \underline{N}_{xz} = \Psi_1, \quad \underline{Q}_x = -F_1', \quad \underline{M}_x = F_1 - G_1\underline{H}/2. \tag{8}$$

Now the elastic constitutive equations of the lower sublaminate have the form

$$\begin{Bmatrix} \underline{N}_x \\ \underline{N}_{xz} \\ \underline{M}_x \end{Bmatrix} = \begin{bmatrix} \underline{A}_{21} & \underline{A}_{22} & \underline{A}_{26} \\ \underline{A}_{61} & \underline{A}_{62} & \underline{A}_{66} \\ \underline{B}_{21} & \underline{B}_{22} & \underline{B}_{26} \end{bmatrix} \begin{Bmatrix} \underline{\varepsilon}_z^0 \\ \underline{\varepsilon}_x^0 \\ \underline{\gamma}_{xz}^0 \end{Bmatrix} + \begin{bmatrix} \underline{B}_{21} & \underline{B}_{22} & \underline{B}_{26} \\ \underline{B}_{61} & \underline{B}_{62} & \underline{B}_{66} \\ \underline{D}_{21} & \underline{D}_{22} & \underline{D}_{26} \end{bmatrix} \begin{Bmatrix} \kappa_z \\ \underline{\kappa}_x \\ 2\Theta \end{Bmatrix}. \tag{9}$$

Eliminating $\underline{N}_x$, $\underline{N}_{xz}$ and $\underline{M}_x$ from the last two sets of equations, and using eqn (6) to express $\underline{\varepsilon}_z^0$, $\underline{\varepsilon}_x^0$ and $\underline{\gamma}_{xz}^0$ in terms of $\varepsilon_{x(1)}$, $\gamma_{xz(1)}$ and $\underline{\kappa}_x$, we obtain a system of linear algebraic equations for $\varepsilon_{x(1)}$, $\gamma_{xz(1)}$ and $\underline{\kappa}_x$

$$\begin{bmatrix} \underline{A}_{22} & \underline{A}_{26} & \underline{B}_{22}+\underline{A}_{22}\underline{H}/2 \\ \underline{A}_{26} & \underline{A}_{66} & \underline{B}_{26}+\underline{A}_{26}\underline{H}/2 \\ \underline{B}_{22} & \underline{B}_{26} & \underline{D}_{22}+\underline{B}_{22}\underline{H}/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{x(1)} \\ \gamma_{xz(1)} \\ \underline{\kappa}_x \end{Bmatrix} = \begin{Bmatrix} G_1 \\ \Psi_1 \\ F_1 - G_1\underline{H}/2 \end{Bmatrix}$$

$$- \begin{bmatrix} \underline{A}_{12} & \underline{B}_{12}+\underline{A}_{12}(t-\underline{H})/2 & \underline{B}_{26}+\underline{A}_{26}\underline{H}/2 \\ \underline{A}_{16} & \underline{B}_{16}+\underline{A}_{16}(t-\underline{H})/2 & \underline{B}_{66}+\underline{A}_{66}\underline{H}/2 \\ \underline{B}_{12} & \underline{D}_{12}+\underline{B}_{12}(t-\underline{H})/2 & \underline{D}_{26}+\underline{B}_{26}\underline{H}/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_0 \\ \kappa_z \\ 2\Theta \end{Bmatrix}. \tag{10}$$

This system of equations may be easily solved to obtain $\varepsilon_{x(1)}$, $\gamma_{xz(1)}$ and $\underline{\kappa}_x$ in terms of F_1, G_1, Ψ_1 and the strain load parameters ε_0, κ_2 and Θ.

Similarly, using the equilibrium equations and the elastic constitutive equations of the upper sublaminate, one may obtain the following system of equations which determines $\varepsilon_{x(3)}$, $\gamma_{xz(3)}$ and $\bar{\kappa}_x$ in terms of F_3, G_3, Ψ_3 and the strain load parameters ε_0, κ_z and Θ

$$\begin{bmatrix} \bar{A}_{22} & \bar{A}_{26} & \bar{B}_{22}-\bar{A}_{22}\bar{H}/2 \\ \bar{A}_{26} & \bar{A}_{66} & \bar{B}_{26}-\bar{A}_{26}\bar{H}/2 \\ \bar{B}_{22} & \bar{B}_{26} & \bar{D}_{22}-\bar{B}_{22}\bar{H}/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{x(3)} \\ \gamma_{xz(3)} \\ \bar{\kappa}_x \end{Bmatrix} = - \begin{Bmatrix} G_3 \\ \Psi_3 \\ F_3 + G_3\bar{H}/2 \end{Bmatrix}$$

$$- \begin{bmatrix} \bar{A}_{12} & \bar{B}_{12}-\bar{A}_{12}(t-\bar{H})/2 & \bar{B}_{26}-\bar{A}_{26}\bar{H}/2 \\ \bar{A}_{16} & \bar{B}_{16}-\bar{A}_{16}(t-\bar{H})/2 & \bar{B}_{66}-\bar{A}_{66}\bar{H}/2 \\ \bar{B}_{12} & \bar{D}_{12}-\bar{B}_{12}(t-\bar{H})/2 & \bar{D}_{26}-\bar{B}_{26}\bar{H}/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_0 \\ \kappa_z \\ 2\Theta \end{Bmatrix}, \tag{11}$$

where $\bar{H} = (t/2) - y_3$ denotes the thickness of the upper sublaminate. We now substitute the solutions of the preceding two systems of equations into the integrals

$$\int (\varepsilon_{x(1)}\delta G_1 + \gamma_{xz(1)}\delta\Psi_1 + \underline{\kappa}_x \delta F_1)\,\mathrm{d}x - \int (\varepsilon_{x(3)}\delta G_3 + \gamma_{xz(3)}\delta\Psi_3 + \bar{\kappa}_x\delta F_3)\,\mathrm{d}x.$$

The results contribute additional terms to the Euler–Lagrange equations associated with the complementary virtual work principle [i.e. the last two integrals of eqn (14a) in the next section].

3. THE COMPLEMENTARY VIRTUAL WORK PRINCIPLE AND THE VARIATIONAL EQUATION

The theoretical analysis of applying the complementary virtual work principle to the sublaminate/layer model differs in certain aspects from the derivation of the variational equations for a pure layer model [as given by eqns (15a, b) of Yin (1991)]. We consider a segment of the laminate of unit length in the axial direction, $0 \leqslant z \leqslant 1$. Equation (12) of Yin (1991), when applied to the two interior layers of the sublaminate/layer model, yields the following expression for the total strain energy of the two layers [instead of eqn (13) of Yin (1991)]

$$\sum \int\int \varepsilon_{ij}\delta\tau_{ij}\,\mathrm{d}x\,\mathrm{d}y - \sum \int\int (\Delta u\delta\tau_{xz} + \Delta v\delta\tau_{yz} + \Delta w\delta\sigma_z)\,\mathrm{d}x\,\mathrm{d}y$$

$$-\int_{\text{upper interface}} (u\delta\tau_{xy} + w\delta\tau_{yz} + v\delta\sigma_y)\,\mathrm{d}x + \int_{\text{lower interface}} (u\delta\tau_{xy} + w\delta\tau_{yz} + v\delta\sigma_y)\,\mathrm{d}x = 0, \quad (12)$$

where the symbol Σ indicates summation over the two interior layers and where

$$\Delta u = -\Theta y + \omega_2, \quad \Delta v = \kappa_z/2 + \Theta x - \omega_1, \quad \Delta w = \varepsilon_0 - y\kappa_z$$
$$\delta\tau_{xz} = \delta\Psi_{,y}, \quad \delta\tau_{yz} = -\delta\Psi_{,x}$$
$$\delta\sigma_z = -(a_{12}\delta F_{,yy} + a_{13}\delta F_{,xx} + a_{16}\delta\Psi_{,y})/a_{11}. \quad (13)$$

In the last expression, a_{ij} represents $\bar{a}_{ij}$ in the upper layer and $\underline{a}_{ij}$ in the lower layer; in the expressions for Δu and Δv, ω_1 and ω_2 represent $\bar{\omega}_1$ and $\bar{\omega}_2$ in the upper layer and $\underline{\omega}_1$ and $\underline{\omega}_2$ in the lower layer.

The two line integrals in eqn (12) contribute the following

$$-\int \{-u_3\delta G_3' - w_3\delta\Psi' + v_3\delta F_3''\}\,\mathrm{d}x + \int \{-u_1\delta G_1' - w_1\delta\Psi' + v_1\delta F_1''\}\,\mathrm{d}x$$

$$= -\int \{\varepsilon_{x(3)}\delta G_3 + (\gamma_{xz(3)} - \bar{\omega}_2)\delta\Psi_3 + \bar{\kappa}_x\delta F_3\}\,\mathrm{d}x$$

$$+\int \{\varepsilon_{x(1)}\delta G_1 + (\gamma_{xz(1)} - \underline{\omega}_2)\delta\Psi_3 + \underline{\kappa}_x\delta \mathrm{F}_1\}\,\mathrm{d}x,$$

where u_i, v_i and w_i ($i = 1, 3$) denote the displacement functions on the ith interface. After performing integration by parts on the second set of double integrals in eqn (12), making use of eqn (13) and the homogeneous boundary conditions of F_i, G_i, Ψ_i, F_i' and G_i' at $x = \pm a$ ($i = 1, 2, 3$), and combining the results with the preceding expressions for the line integrals along the interfaces, one obtains from eqn (12) the following variational equation

$$\sum\left[\int\int \varepsilon_{ij}\delta\sigma_{ij}\,\mathrm{d}x\,\mathrm{d}y + \int\int (\kappa_z a_{16}/a_{11} - 2\Theta)\delta\Psi\,\mathrm{d}x\,\mathrm{d}y\right]$$

$$-(\varepsilon_0 - y_2\kappa_z)\left\{(\bar{a}_{12}/\bar{a}_{11} - \underline{a}_{12}/\underline{a}_{11})\int \delta G_2\,\mathrm{d}x + (\bar{a}_{16}/\bar{a}_{11} - \underline{a}_{16}/a_{11})\int \delta\Psi_2\,\mathrm{d}x\right\}$$

$$+(\varepsilon_0 - y_3\kappa_z)\left\{(\bar{a}_{12}/\bar{a}_{11})\int \delta G_3\,\mathrm{d}x + (\bar{a}_{16}/\bar{a}_{11})\int \delta\Psi_3\,\mathrm{d}x\right\}$$

$$-(\varepsilon_0 - y_1\kappa_z)\left\{(\underline{a}_{12}/\underline{a}_{11})\int \delta G_1\,\mathrm{d}x + (\underline{a}_{16}/\underline{a}_{11})\int \delta\Psi_1\,\mathrm{d}x\right\}$$

$$-\kappa_z(\bar{a}_{16}/\bar{a}_{11} - \underline{a}_{16}/\underline{a}_{11})\int \delta F_2\,\mathrm{d}x + \kappa_z(\bar{a}_{16}/\bar{a}_{11})\int \delta F_3\,\mathrm{d}x - \kappa_z(\underline{a}_{16}/\underline{a}_{11})\int \delta F_1\,\mathrm{d}x$$

$$+\int (\varepsilon_{x(1)}\delta G_1 + \gamma_{xz(1)}\delta\Psi_1 + \underline{\kappa}_x\delta F_1)\,\mathrm{d}x - \int (\varepsilon_{x(3)}\delta G_3 + \gamma_{xz(3)}\delta\Psi_3 + \bar{\kappa}_x\delta F_3)\,\mathrm{d}x = 0, \quad (14a)$$

where

$$\sum \iint \varepsilon_{ij}\delta\sigma_{ij}\,\mathrm{d}x\,\mathrm{d}y = \sum \iint \{F_{,yy}, F_{,xx}, -F_{,xy}, -\Psi_{,x}, \Psi_{,y}\}$$

$$\times \begin{bmatrix} \beta_{22} & \beta_{23} & 0 & 0 & \beta_{26} \\ \beta_{23} & \beta_{33} & 0 & 0 & \beta_{36} \\ 0 & 0 & \beta_{44} & \beta_{45} & 0 \\ 0 & 0 & \beta_{45} & \beta_{55} & 0 \\ \beta_{26} & \beta_{36} & 0 & 0 & \beta_{66} \end{bmatrix} \begin{Bmatrix} \delta F_{,yy} \\ \delta F_{,xx} \\ -\delta F_{,xy} \\ -\delta\Psi_{,x} \\ \delta\Psi_{,y} \end{Bmatrix} \mathrm{d}x\,\mathrm{d}y. \quad (14b)$$

In eqn (14a), only the last two line integrals depend on the stiffness matrices of the sublaminates through the expressions of eqns (10) and (11) for $\varepsilon_{x(1)}$, $\gamma_{xz(1)}$, $\underline{\kappa}_x$, $\varepsilon_{x(3)}$, $\gamma_{xz(3)}$, and $\bar{\kappa}_x$ in terms of F_1, G_1, Ψ_1, F_3, G_3, Ψ_3 and the strain load parameters ε_0, κ_z and Θ. All other terms of the variational equation are contributed by the two interior layers, and therefore depend only on the strain load and the elastic properties of the interior layers.

4. POLYNOMIAL APPROXIMATIONS OF THE STRESS FUNCTIONS

If the stress functions $\underline{F}$ and $\underline{\Psi}$ in the lower layer are approximated, respectively, by cubic and quadratic functions of the normalized thickness coordinate $\eta = (y-y_1)/(y_2-y_1)$, then

$$\underline{F}(x,\eta) = (1-3\eta^2+2\eta^3)F_1(x)+(3\eta^2-2\eta^3)F_2(x) + (\eta-2\eta^2+\eta^3)\underline{h}G_1(x)+(-\eta^2+\eta^3)\underline{h}G_2(x) \quad (15)$$

$$\underline{\Psi}(x,\eta) = (1-\eta^2)\Psi_1(x)+\eta^2\Psi_2(x)+(\eta-\eta^2)\underline{h}H_1(x), \quad (16)$$

where $\underline{h} = y_2-y_1$ is the thickness of the lower layer and where $H_1(x)$ is the value of $\partial\underline{\Psi}/\partial y = \underline{\Psi}_{,\eta}/\underline{h}$ on the interface $y = y_1$. Analogous polynomial expansions may be given for the stress functions $\bar{F}$ and $\bar{\Psi}$ in the upper layer in terms of the normalized thickness coordinate $\eta = (y-y_2)/(y_3-y_2)$, in which the coefficient functions F_i, G_i, Ψ_i and H_i are replaced, respectively, by F_{i+1}, G_{i+1}, Ψ_{i+1} and H_{i+1} (where H_2 is the value of $\partial\bar{\Psi}/\partial y$ on the interface $y = y_2$).

Substituting the preceding polynomial expansions of the stress functions into eqns (14a) and (14b), and following the procedure described in Section 7 of Yin (1991), we obtain a system of ordinary differential equations with constant coefficients

$$([\mathbf{W}]\,\mathrm{d}^4/\mathrm{d}x^4+[\mathbf{V}]\,\mathrm{d}^2/\mathrm{d}x^2+[\mathbf{U}])\{\mathbf{X}\} = \{\mathbf{b}\}, \quad (17)$$

where [W], [V] and [U] are 11×11 symmetric matrices and the nonzero elements of [W] appear only in the 6×6 submatrix at the upper left corner. The elements of the column vector {X} are the 11 unknown functions F_1, F_2, F_3, G_1, G_2, G_3, Ψ_1, Ψ_2, Ψ_3, H_1 and H_2, which satisfy the following homogeneous boundary conditions at the free edges

$$F_i(\pm a) = G_i(\pm a) = \Psi_i(\pm a) = F_i'(\pm a) = G_i'(\pm a) = 0, \quad i = 1,2,3$$
$$H^{(1)}(\pm a) = H^{(2)}(\pm a) = 0. \quad (18)$$

The first two symmetric matrices [W] and [V] depend only on the thicknesses and anisotropic elastic properties of the two interior layers, while the third matrix [U] and the constant vector {b} depend also on the thicknesses and the stiffness properties of the sublaminates. The characteristic equation associated with eqn (17)

$$\text{Determinant }([\mathbf{W}]\lambda^4+[\mathbf{V}]\lambda^2+[\mathbf{U}]) = 0 \quad (19)$$

is a polynomial equation of degree 17 in λ^2. Hence there are 17 pairs of real and complex eigenvalues λ. Each pair consists of eigenvalues which differ only in algebraic sign. A particular solution of the differential equation (17) is

$$\{\mathbf{X}\}_p = [\mathbf{U}]^{-1}\{\mathbf{b}\}$$

which depends on the three strain load parameters ε_0, κ_z, and Θ through the constant vector $\{\mathbf{b}\}$. The unique solution of the eigenvalue problem may be obtained by taking an appropriate linear combination of the eigenfunctions with the preceding particular solution so that the 34 homogeneous boundary conditions of eqn (18) are all satisfied.

If the interface under consideration, $y = y_2$, is the lowest interface in the laminate, then the sublaminate/layer model has no lower sublaminate and the functions $F_1(x)$, $G_1(x)$ and $\Psi_1(x)$ vanish identically. The dimension of the vectors $\{\mathbf{X}\}$ and $\{\mathbf{b}\}$ is diminished from 11 to 8 and the characteristic equation, eqn (19), reduces to a polynomial equation of degree 12 in λ^2. A similar situation occurs when $y = y_2$ is the highest interface of the laminate.

Another degenerate case is associated with a cross-ply laminate having the material axes parallel and perpendicular to the free edges. In this case the solution space of the eigenvalue problem decomposes orthogonally into two subspaces associated, respectively, with the stress functions F and Ψ. The first subspace determines the solutions under axial extension ε_0 and bending κ_z while the second subspace determines the solutions under the twisting deformation Θ.

The algebraic manipulations leading from the variational equation, eqn (14), and the polynomial approximations, eqns (15) and (16), to the system of ordinary differential equations, eqn (17), is exceedingly laborious and cannot be achieved without using computer algebra. The symbolic algebraic program MACSYMA is used to obtain the explicit expressions of all elements of the symmetric matrices [W], [V] and [U] in terms of the geometry and stiffness of the interior layers and the sublaminates, and to obtain the vector $\{\mathbf{b}\}$ in terms of these data and the strain load parameters ε_0, κ_z and Θ. These explicit expressions form the basis of a FORTRAN program which evaluates the symmetric matrices [W], [V] and [U] and the vector $\{\mathbf{b}\}$ by using the input data concerning the material properties, the thicknesses and the orientation angles of the successive layers, and the width of the laminated strip. A subsequent segment of the FORTRAN program then computes all eigenvalues of the characteristic equation, eqn (19), and the associated eigenvectors. Finally, the solutions for the unknown functions $\{\mathbf{X}\}$ are obtained for each one of the three loading cases by suitably combining the eigenfunctions with the relevant particular solution (Yin, 1990).

The FORTRAN program has built-in subroutines to generate the elastic compliance matrices ($[a_{ij}]$ and $[\beta_{ij}]$) of the interior layers and the stiffness matrices ($[A_{ij}]$, $[B_{ij}]$ and $[D_{ij}]$) of the sublaminates. Furthermore, the two degenerate cases—one corresponding to an orthogonally decomposable solution space (cross-ply laminates) and the other concerning the absence of one sublaminate—have been included within the capability of the program. Thus, although the symbolic algebra leading to the development of the FORTRAN program is exceedingly complex, the resulting program is extremely easy to use because its implementation does not involve tedious geometrical modeling tasks (such as mesh generation in a finite element analysis). The program is applicable to a laminate composed of any number of differently oriented plies with arbitrary orthotropic elastic properties. A single execution of the program for evaluating the interlaminar stresses on a particular interface in a multilayer laminate takes only seconds on a 386 or 486 personal computer. The reliability and accuracy of the program are demonstrated in the following section by the solutions for symmetric, four-layer, cross-ply, and angle-ply laminates.

5. ANALYTICAL RESULTS AND VALIDATION

The classical problems of symmetric, four-layer, cross-ply, and angle-ply laminates have been extensively studied and many numerical solutions of the interlaminar stresses have been presented by various authors for the particular case of the axial strain load. The interlaminar stresses determined by the stress-function-based variational method (Yin,

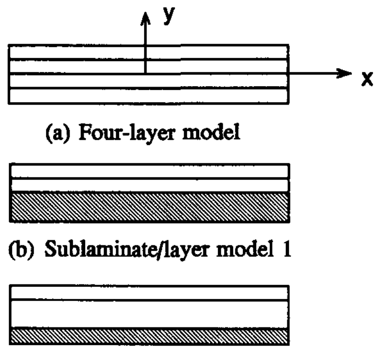

Fig. 2. Three analytical models of a four-layer laminate.

1991), using cubic polynomial expansions for F and quadratic expansions for Ψ, were found to be in reasonable agreement with some of the most refined finite-element solutions in the existing literature. In the present work, we apply the sublaminate/layer approach to the same four-layer laminates and compare the resulting interlaminar stresses for the axial strain and twisting loads with the corresponding variational solutions based on a pure layer model.

We consider four-layer laminates made of identical unidirectional plies whose orthotropic elastic moduli are as given in Wang and Crossman (1977). The distance between the free edges is 16 times the ply thickness, i.e. $2a = 16h$. In addition to the four-layer analytical model [used previously in Yin (1991)], two sublaminate/layer models may be used to calculate the interlaminar stresses across the highest interface $y = h$. In sublaminate/layer model 1 (SL1), stress functions are introduced in the two upper layers, while the two lower layers are considered as the constituent layers of the lower sublaminate. In sublaminate/layer model 2 (SL2), the two middle layers of the laminate which have the same ply orientation is considered as a single layer of thickness $2h$ and the lowest layer of the laminate is taken to be a single-layer sublaminate. In both models there is no upper sublaminate. All three analytical models are shown in Fig. 2.

For the symmetric angle-ply laminate ($[45/-45]_s$) under a unit axial strain load, $\varepsilon_0 = 1$, the results for the normal and shearing stresses on a cross-section of the laminate, z = constant, are shown in Fig. 3. The results for the interlaminar stresses on the $45/-45$ interface ($y = h$) are shown in Fig. 4. In these figures, the origin of the x-coordinate has been shifted so that the free edges are located at $x = 0$ and $x = 16h$. While the results from all three analytical models are reasonably close, the agreement between the solutions of the SL2 model and the four-layer model is especially good except in a short interval adjacent to the free edge. Significant discrepancies are noticed only in the interlaminar stresses σ_y and τ_{xy}, which are much smaller in magnitude than the dominant interlaminar stress τ_{yz}.

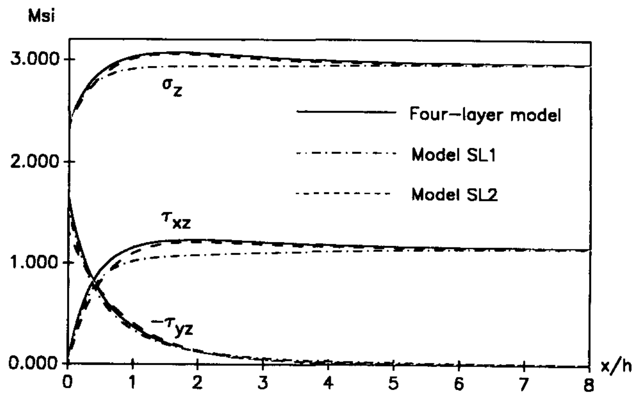

Fig. 3. σ_z, τ_{xz} and τ_{yz} in a $[45/-45]_s$ laminate under unit axial strain.

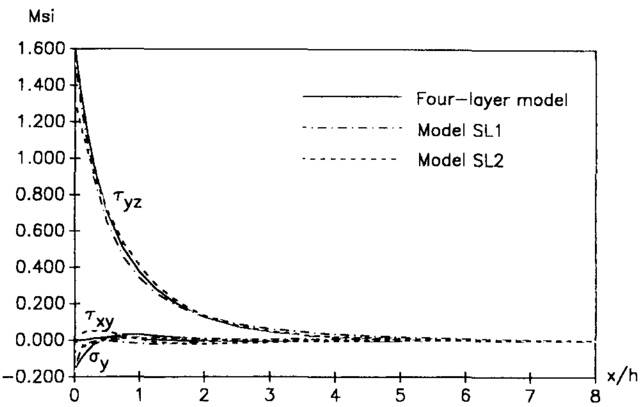

Fig. 4. Interlaminar stresses in a $[45/-45]_s$ laminate under unit axial strain.

Under a strain load characterized by the bending curvature κ_z, the material region near the interface $y = h$ is subjected, approximately, to an axial compressive strain $-\varepsilon_z = \kappa_z h$. Hence the interlaminar stresses on that interface are nearly identical to those produced by a uniform axial strain $\varepsilon_0 = -\kappa_z h$ in the laminate. The results obtained from the present solutions are found to be in good agreement with the finite-element solutions given recently by Ye (1990).

For the strain load corresponding to a unit twisting deformation, $\Theta = 1/h$, the results for the interlaminar stresses on the $45/-45$ interface are shown in Fig. 5. The results for the four-layer model and for the SL2 model are exceedingly close, while the interlaminar shearing stresses τ_{yz} and τ_{xy} computed by the SL1 model are, in comparison to the above results, somewhat greater near the free edge and smaller near the center line of the laminate. This happens because, by combining the two lower layers with significantly different orientation angles (45° and −45°) into a sublaminate and imposing the kinematical assumption of the classical plate theory (i.e. plane sections of the sublaminate remain plane and perpendicular to the middle surface), the SL1 model overestimates the stiffness in the regions adjacent to the free edges where the deviation of the actual deformation of the sublaminate from the Kirchhoff–Love assumption is more prominent. Figures 4 and 5 show that, for the $[45/-45]_s$ laminate under all three types of strain loads (axial strain, bending, and twisting), the mode III interlaminar stress τ_{yz} dominates over the peeling stress σ_y and the mode II shearing stress τ_{xy}.

As pointed out in Yin (1991), along the interface $y = y_2$, the values of the functions F'_2, $-G'_2$ and Ψ_2 are equal to the resultant forces of the interlaminar stresses σ_y, τ_{xy} and τ_{yz}, respectively, over the end interval $[0, x]$ of the interface. These functions indicate the resultant peeling and (mode I and mode II) shearing actions acting across the end segment of the interface. They are useful measures of the criticality of the interlaminar action. Figures 6 and 7 show the plots of these functions for the $[45/-45]_s$ laminate subjected to

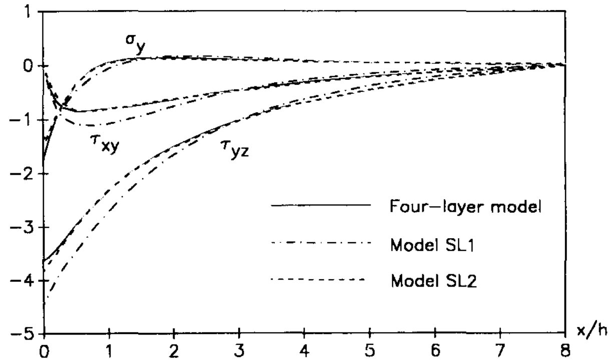

Fig. 5. Interlaminar stresses in a $[45/-45]_s$ laminate under twisting ($\Theta = 1/h$).

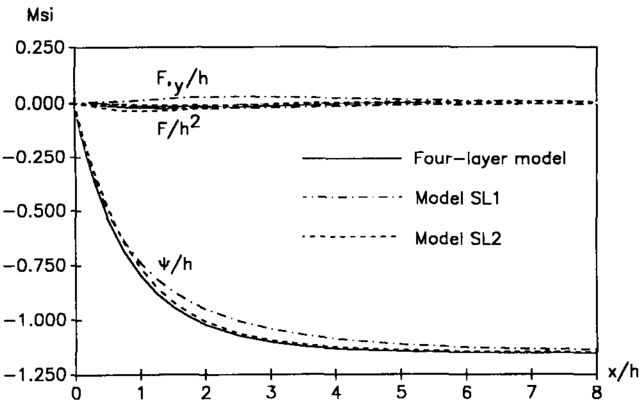

Fig. 6. F, $F_{,y}$ and Ψ in a $[45/-45]_s$ laminate under unit axial strain.

$\varepsilon_0 = 1$ and $\Theta = 1/h$, respectively. The two plots indicate that, under both types of strain load, the dominant (mode III) shearing action is concentrated in a short interval adjacent to the free edge, since the function value of Ψ_2 increases rapidly from zero at $x = 0$ and approaches the maximum value over a short end interval of length comparable to the laminate thickness. Furthermore, while the three analytical models yield significantly different pointwise values of the interlaminar stresses in an immediate vicinity of the free edge (where the interlaminar stresses may approach infinite values according to the singularity solution of the elasticity theory), they yield resultant peeling and shearing forces that are in much better agreement. This suggests that, although a variational method of analysis using the sublaminate/layer approach may yield poor results for the interlaminar stress near the free edge, the analysis generally provides accurate results for the resultant forces over short end segments of the interface—quantities that are especially valuable as measures of the criticality of interlaminar action because, compared to the pointwise values of the interlaminar stresses, the stress resultants are less sensitively affected by deviation of the material behavior from the idealized assumption of linear elasticity.

In the case of cross-ply laminates, the differential equations for the stress functions F and Ψ are uncoupled. The function Ψ vanishes identically under the axial strain and bending deformation, while F vanishes identically under the twisting deformation. For two types of symmetric, four-layer, cross-ply laminates with the stacking sequences $[0/90]_s$ and $[90/0]_s$, the interlaminar stresses on the interface $y = h$ under a unit axial strain load are shown in Fig. 8 for σ_y and in Fig. 9 for τ_{xy}. Figures 10 and 11 show, respectively, the resultant peeling and shearing forces (F'_2 and $-G'_2$) acting across end segments of the interface of varying lengths. Figure 12 shows the interlaminar shearing stress τ_{yz} on the same interface under a unit twisting deformation. The results are identical for the $[0/90]_s$ and $[90/0]_s$ laminates. Also shown in the figure is the resultant shearing force across a variables end segment $[0, x]$ of the interface, which is given by the function Ψ_2. Under both the axial strain load and

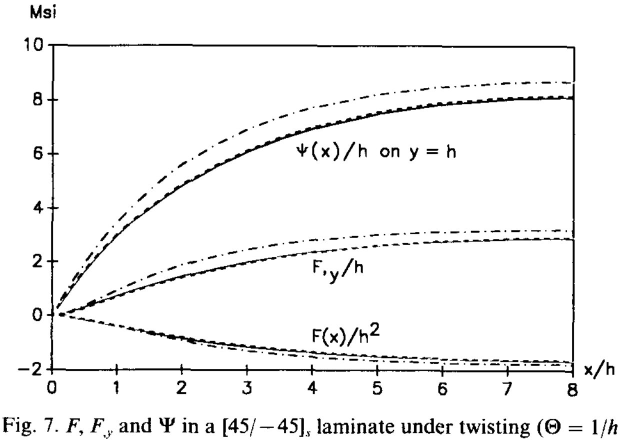

Fig. 7. F, $F_{,y}$ and Ψ in a $[45/-45]_s$ laminate under twisting ($\Theta = 1/h$).

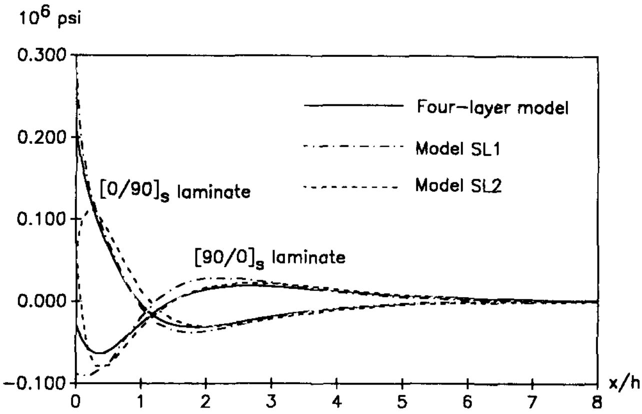

Fig. 8. σ_y on $y = h$ in cross-ply laminates under unit axial strain.

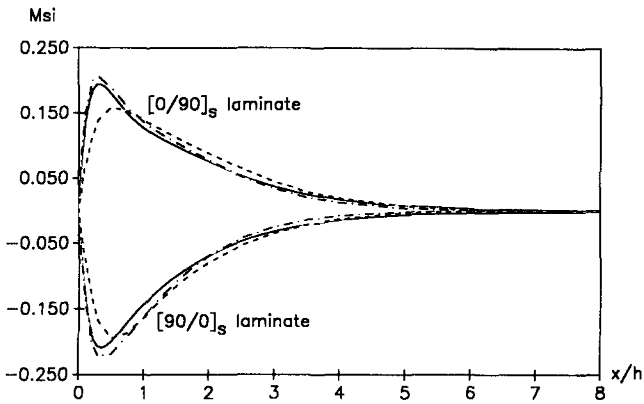

Fig. 9. τ_{xy} on $y = h$ in cross-ply laminates under unit axial strain.

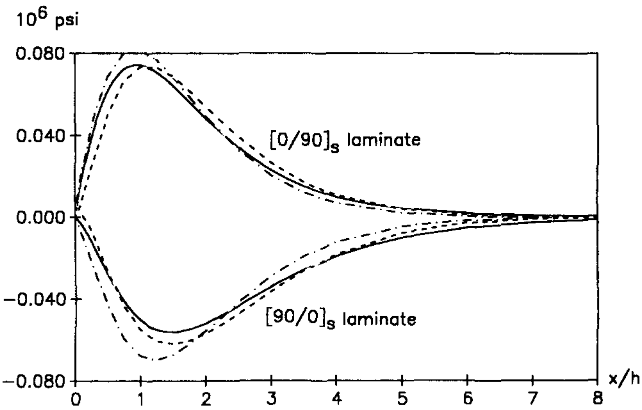

Fig. 10. F_x on $y = h$ in cross-ply laminates under unit axial strain.

the twisting load, the results computed from the four-layer model and two sublaminate/layer models once again show superior agreement in the resultant peeling and shearing forces and less satisfactory agreement in the pointwise values of the interlaminar stresses in an immediate vicinity of the free edge.

Wang and Crossman (1977) calculated the interlaminar stresses in two types of symmetric, eight-layer, quasi-isotropic laminates ($[45/-45/0/90]_s$ and $[90/0/-45/45]_s$) under an axial strain load by using the finite element method. In the present work, variational solutions for the interlaminar stresses in the same laminates are obtained by repeated use of the sublaminate/layer model (one model for each interface). Each solution step yields the interlaminar stresses on a particular interface for all three cases of the strain load.

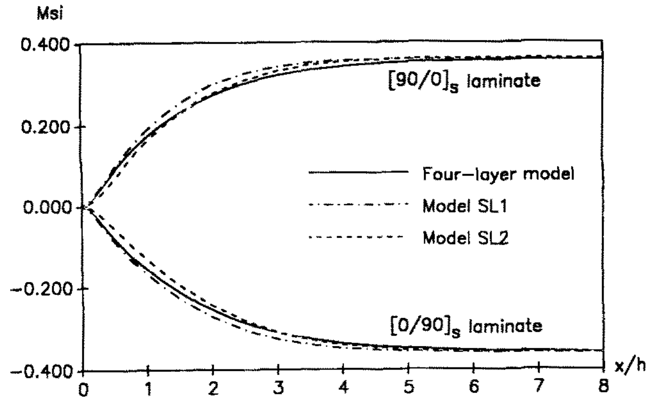

Fig. 11. $F_{,y}$ on $y = h$ in cross-ply laminates under unit axial strain.

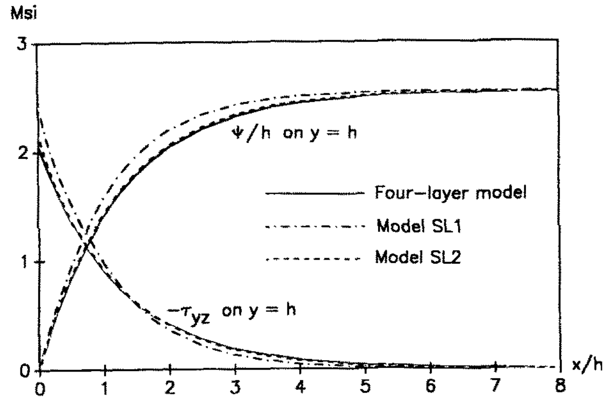

Fig. 12. $-\tau_{yz}$ and Ψ/h in cross-ply laminates under twisting ($\Theta = 1/h$).

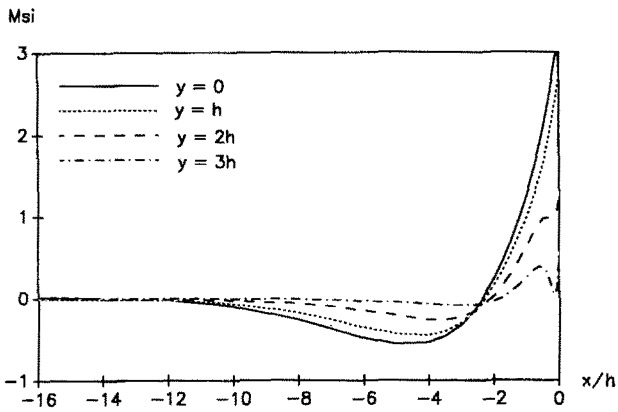

Fig. 13. σ_y in a $[45/-45/0/90]_s$ laminate under unit axial strain.

For the $[45/-45/0/90]_s$ laminate under a unit axial strain load, the interlaminar stresses on the various interfaces are shown in Figs 13–15. The present results are in reasonable agreement with Wang and Crossman's finite element solutions as shown in their Figs 11 and 12, except in short intervals of the interfaces adjacent to the free edge. Notice that a different coordinate system was used by Wang and Crossman (1977). In Figs 13–15, we show the interlaminar stresses across the *right* half of the interface so that the present results for τ_{xy}, τ_{yz} and σ_y, respectively, may be easily compared with their results for τ_{yz} (shown for the interface $y = h$ only), τ_{xz} (for the interface $y = 3h$) and σ_z (for all four interfaces).

For the $[90/0/-45/45]_s$ laminate under the axial strain load, and for both types of quasi-isotropic laminates subjected to bending and twisting deformation, the interlaminar stresses on the successive interfaces have also been obtained (Yin, 1992a; see Figs 14, 16

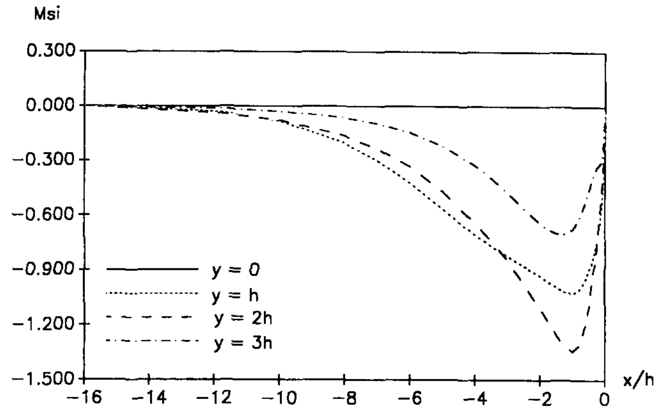

Fig. 14. τ_{xy} in a $[45/-45/0/90]_s$ laminate under unit axial strain.

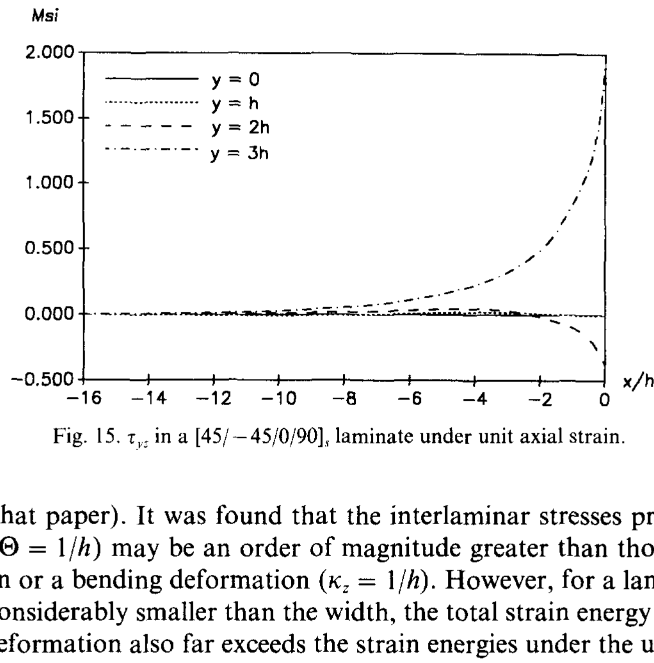

Fig. 15. τ_{yz} in a $[45/-45/0/90]_s$ laminate under unit axial strain.

and 18–22 of that paper). It was found that the interlaminar stresses produced by a unit twisting load ($\Theta = 1/h$) may be an order of magnitude greater than those produced by a unit axial strain or a bending deformation ($\kappa_z = 1/h$). However, for a laminated strip with the thickness considerably smaller than the width, the total strain energy associated with a unit twisting deformation also far exceeds the strain energies under the unit axial strain or unit bending deformation.

6. SUMMARY AND CONCLUDING REMARKS

A sublaminate/layer model is used to investigate the interlaminar stresses near the free edge of a multi-layered laminate subjected to three distinct cases of strain loads, i.e. axial extension, bending and twisting. Using the stiffness matrices and the equilibrium equations of the sublaminates, the kinematical variables of the sublaminates are expressed in terms of the Lekhnitskii stress functions in the two interior layers. This yields a variational problem for the sublaminate/layer model in a purely stress formulation, based on the principle of complementary virtual work. The stress functions in each layer are expanded in truncated power series of the thickness coordinate, and the coefficient functions of the expansion are determined from the solution of an eigenvalue problem. The sublaminate/layer approach limits the dimension of the eigenvalue problem to a fixed number irrespective of the numbers of layers in the two sublaminates, so that reasonably accurate solutions of the interlaminar stresses can be obtained with little computational effort. While the solutions satisfy strain compatibility and interfacial continuity of displacements only in the sense of the mean, they satisfy *exactly* the equilibrium equations of the layers and the sublaminates, traction-free boundary conditions at the free edge, and continuity of interlaminar stresses across the interface.

The prominent features of the present analysis method are its generality and efficiency. The method is applicable to a laminate composed of any number of layers with arbitrary orientations and anisotropic (linearly) elastic properties. A FORTRAN program is developed for implementing the analysis on personal computers. Execution of the program requires only the numerical data of the elastic moduli, orientation angles, and thicknesses of all layers and the width of the laminated strip. No geometrical modeling task such as mesh generation is needed. For a given interface in a laminate, interlaminar stresses for all three cases of strain load—axial extension, bending, and twisting—are computed in the same eigenfunction analysis. Thermal stress analysis for a given temperature load (which may depend on the thickness coordinate) can also be performed using the procedure described in Yin (1993). The program is suitable for repeated use in parametric studies, design and optimization of composite laminates to control interlaminar stresses under various states of loading.

The sublaminate/layer method is applied to symmetric, four-layer, cross-ply and $\pm 45°$ angle-ply laminates and to two types of symmetric, eight-layer quasi-isotropic laminates. The present solutions are compared with the existing finite element solutions in the literature for the special case of the axial strain load, and to previous stress-function-based variational solutions without the use of sublaminates. The agreement is found to be practically acceptable, particularly for the dominant component of the interlaminar stress. Furthermore, the different solutions show better agreement in the resultant forces of the interlaminar peeling and shearing stresses over end segments of the interface. Further improvement in the accuracy of the variational solutions may be achieved by using higher-degree polynomial expansions of the stress functions in the two interior layers in conjunction with a more refined laminated plate theory for the sublaminates. Alternatively, the present variational method may be implemented twice to obtain preliminary solutions for the interlaminar stress on the $(i-1)$th and $(i+1)$th interfaces. In the segments of the interfaces immediately adjacent to the free edge, these solutions may deviate significantly from the corresponding elasticity solutions in a pointwise sense. However, they are expected to agree closely with the latter solutions in regard to the various moments

$$\left(\int x^n \sigma_y \,\mathrm{d}x, \quad \int x^n \tau_{xy} \,\mathrm{d}x, \quad \text{and} \quad \int x^n \tau_{yz} \,\mathrm{d}x, \quad n = 0, 1, \ldots\right).$$

By virtue of Saint Venant's principle, the variational solutions of the interlaminar stresses on the $(i-1)$th and $(i+1)$th interfaces should serve very well as the traction boundary conditions of a boundary-interface problem including only two layers adjacent to the ith interface. More accurate solution procedures with the consideration of stress singularity may be applied subsequently to the two layer problem. This two-step solution scheme provides a highly efficient procedure for accurate solution of the interlaminar stresses on the ith interface.

Although the present analysis assumes that the strain and curvature loads ε_0, κ_z and Θ are constant in the laminate, this assumption may be weakened to the extent that the load parameters vary slowly in the plane of the laminate. The intense and localized interlaminar stresses near a particular location of the free edge depend mainly on the *local* values of the load parameters. For, just as the interlaminar stresses decay rapidly away from the free edge and affect very little the stress state in the interior region of the laminate, so the interlaminar stresses near the free edge are not significantly affected by moderate deviations of the strain and curvature loads from their values at the free edge. This heuristic argument, relevant to many problems involving stress concentration in narrow boundary regions, is suggested by Saint Venant's principle and the reciprocity relations of elasticity.

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